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Computer Determination of the Constituent Structure of Biological Images*

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A class of algorithms is described which enables computer quantized images to be decomposed into constituent parts reflecting the structure of the images. This decomposition is viewed as the morphological precursor to a higher level syntactic analysis. Numerical results for a typical biological image are presented.

INTRODUCTION

Digitized pictures can be processed with computers for any one of several fundamentally different purposes. One such purpose exemplified by image enhancement (*1*) is typical in that it has as its output a picture derived from an input one. In such applications the fundamental purpose is the transformation of one image into another which is primarily intended for visual consumption by people. A second purpose involves essentially artificial or synthetic images. Although these data do derive from the natural world in a certain sense, they are produced in situations under sufficient human or mechanical control so that their artificial, schematic, and synthetic aspects can be heavily exploited in analysis (*2*). A third purpose served by information processing is to produce a single datum, usually the one which identifies the name of a pattern or the class to which it belongs. This case may be considered to be a special instance of image processing problems in which the purpose served by processing a pictorial image is to obtain a data structure from that image—one which may be a description of the image or a partial description of the image in association with other nonimage data. It is this class of purposes which concerns us in this paper, and the particular type of data structure that we are interested in obtaining from an image is one which describes how the image is constituted out of its component parts. We are thus concerned here with pattern recognition in which recognition is understood in the wider sense of not only the naming of a pattern but also the naming of its structural parts with an indication also of their relations to each other. We will be concerned here with algorithmic methods for determining the structural decomposition of an image into its constituent parts.

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I. Sources of Decomposition of Images

A. *Intrinsic Decomposition*

If we consider images independent of their particular digitized or other form of quantized representation in a computer, it will be noted that such images typically can be partitioned into component parts by making use of data which are intrinsic to the images themselves. Thus, the image which constitutes this printed page can easily be seen to be partitioned into subparts consisting of the individual letters and other orthographic elements on the page. The criterion for this partitioning is largely inherent in the image itself, since the boundaries between letters and the background of the page is well defined enough to be exploitable for purposes of isolating the printed characters. In natural photographs, however, the situation is somewhat more complex because there are usually at least two classes of objects in natural photographs, those which have well-defined boundaries, and those which can only be looked at as statistical aggregates.

1. *Well-defined objects.* When a photograph or other image source comes from an object which consists of well-defined subobjects, very often the definition of the subobjects and the consequent partitioning of the pictorial representation can be obtained by invoking some simple criterion. Two such criteria are the use of a gradient function to determine boundaries or the use of a thresholding mechanism which establishes boundaries at a place where the optical information function exceeds some prescribed threshold. These methods work, however, only where the image comes from an object which is well-defined in terms of its subobjects.

2. *Aggregates.* An important class of natural images fails to satisfy the above criterion. Here the objects in an image are aggregates of smaller objects. The smaller objects need not be distinguishable or individually identifiable, but their aggregative nature is identifiable. We are thus drawing a distinction between the forest and the trees in suggesting that different methods are usable for determining the boundaries of the forest from those which can determine the boundaries of trees. In biological examples, we have cases like cells with well-defined membranes, nuclei of cells which also have well-defined borders in suitably stained preparations, and, by way of contrast, regions in tissue sections which are defined more by densities of cellular constituents than by sharply defined boundaries, or certain substructures like nucleoli or nissl flakes in nerve cells. There are methods for establishing the boundaries of aggregative objects, but they differ from the methods that work with well-defined objects. Mendelsohn, et al. (*3*) present a method which works for establishing the boundaries of objects like chromosomes whose images should properly be viewed as aggregates.

We will present below an algorithm that exploits the interference between separate objects to establish boundaries not only between well-defined objects but also between aggregative objects.

B. Decomposition by Externally Imposed Syntax

Although we have suggested above that various kinds of measurements performed on patterns can yield recognition of their component parts, it is useful to notice that component parts of an image can, in certain cases, be identified without making any measurements at all upon the components. In the case of highly structured images (again using the example of this printed page), the degree of redundancy in the structure is such as to make it possible to recognize certain component parts once we know two facts: first, the context in which the components exist and, second, the mere fact of existence of those components. The problem of recognition, then, becomes transformed into a problem of recognition of the context and mere detection of the existence of a component in an image. It is thus the syntax, the higher order structural organization of the image that enables the recognition of certain of the component parts of that image.

The farther we get, however, from highly stylized and therefore artificial image sources, the less likely are we to be able to exploit the syntactic structure of these images for purposes of recognizing their component parts. In the case of natural images, such as occur in most biological applications of image processing, there is, nevertheless, a certain degree of residual redundancy in the images which can be exploited in recognizing component structure. However, in most of these cases the redundancy serves more to resolve ambiguities in the recognition of component parts than it does to determine the recognition in its entirety. We would expect, therefore, in biological images, to find that the externally imposed syntax with which a particular image may be viewed can serve to resolve morphological ambiguities where precise measurements on the component parts yield only ambiguous conclusions with respect to the identity of the morphological parts.

The overall conclusion that we must draw from the above discussion with respect to image processing for natural biological subjects is that the recognition of constituent structure for such images must result from an interplay between the morphological identification of the components and the invoking of an *a priori* syntactical structure imposed upon the image from knowledge that derives from sources other than the image itself. And we would expect that this interplay will be such that during the recognition process both syntactical and morphological recognition criteria will be used to invoke the other kind in a fairly complex kind of recognition algorithm. The morphological part of such a recognition algorithm is described in this paper.

II. A Class of Decomposition Algorithms

A. A Particular Example

We wish now to consider a particular example of how the decomposition at the morphological level may be made for a gray scale image that is prepared for computer input. The type of data to which such an analysis is applicable is best illustrated by the description given by Stein, Lipkin, and Shapiro (*4*) in which the method for obtaining such data with a computer controlled microscope is described. We will first consider a particular example of how one may do morphological analysis, and then we will generalize to include a wide variety of different cases.

The picture which is to be analyzed may be represented, as in Fig. 1, as a rectangular array of 36 × 36 decimal numbers. These quantities represent the brightness of corresponding elementary regions in the original picture. We will not specify whether the image is considered to consist of dark objects on a light background or conversely, since the method of analysis at the morphological level should be equally

2	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	2	2	1	0	1	1	1	1	1	1	1	2	2	1	1	1	1	1	2	0
2	2	2	2	2	1	1	1	1	1	1	0	0	1	1	2	1	1	1	1	1	0	1	2	2	1	2	2	2	2	1	1	1	1	1	1
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1	1	1	1	2	2	2	2	2	1	1	2	2	1	2	2	1	2	2	2	2	2	3	4	6	8	9	9	8	9	7	4	3	3	2	2
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1	2	2	2	3	4	6	8	10	11	12	12	10	7	5	4	3	4	5	6	8	8	11	10	9	8	8	10	11	11	8	5	3	2	2	2
1	2	2	2	2	3	4	6	7	10	12	12	9	6	3	3	3	3	4	6	9	11	11	8	8	8	9	12	13	10	7	4	2	2	1	2
1	1	1	2	2	2	3	5	6	9	11	11	9	5	3	3	2	3	4	6	8	11	11	10	8	8	11	12	12	10	7	4	3	2	2	2
1	1	2	2	2	3	3	3	4	7	9	10	8	5	3	3	3	2	3	5	8	11	11	8	8	7	9	11	11	9	6	4	2	2	2	3
2	2	2	2	2	2	2	2	4	6	7	7	6	4	3	2	2	2	3	5	6	9	10	9	7	6	7	9	10	9	6	3	2	3	2	2
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2	2	2	2	2	2	2	2	2	4	4	3	3	3	2	1	1	1	2	3	4	5	6	5	5	4	4	5	5	5	4	2	2	2	2	1
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2	1	2	2	1	2	2	2	2	2	2	2	3	2	1	1	2	2	2	2	2	2	2	2	2	1	1	2	2	2	2	1	1	2	2	2

FIG. 1. A 36 × 36 array of (decimal) brightness values.

applicable to figure and ground, treating both of these symmetrically. Within the input image (thus quantized), we wish to identify regions with varying degrees of heterogeneity of brightness values. These regions and a suitable structure imposed upon them will represent the output of the morphological analysis process.

For the purpose of our example we will choose a particular contrast function in the following manner: at any arbitrary point p, as shown in Fig. 2, we will number the eight adjacent points as $a_0, a_1, \ldots, a_7$ and then evaluate the contrast function:

$$\max\left[1, \max_{i=0}^{7} |5(a_i + a_{i+1} + a_{i+2}) - 3(a_{i+3} + a_{i+4} + \cdots + a_{i+7})|\right]$$

where the subscripts are evaluated modulo 8. It will be seen that this nonisotropic function is related to the magnitude of the gradient of the original brightness

$$\begin{matrix} a_0 & a_1 & a_2 \\ a_7 & p & a_3 \\ a_6 & a_5 & a_4 \end{matrix}$$

Numbering of elements in the neighborhood of an arbitrary point, p

Contrast function at p

$$= \max\left[1, \max_{i=0}^{7} |5(a_i + a_{i+1} + a_{i+2}) - 3(a_{i+3} + \cdots + a_{i+7})|\right]$$

Where all subscripts are evaluated modulo 8.

FIG. 2. A brightness contrast function.

function. It is nonsymmetric and sensitive to small changes in the value of the gradient. If we perform this computation homogeneously for the brightness function in Fig. 1, the result is as shown in Fig. 3, where we have chosen to use two decimal characters to represent the magnitude of the contrast function at every point.

We will say that point p in Fig. 2 is *adjacent* to points a_0 through a_7 and we will speak of a *blob* as a set of points, each of which may be reached from all the others in the set by moving from a point to an adjacent point in the set. We may finally introduce the notion of a blob of heterogeneity K by making simultaneous use of the notion of a blob and of the contrast function previously calculated. By a *blob of heterogeneity K* we will mean a set of points in the original brightness function

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211812181006101212150813100810100906050608111009120410110511101510010515
152012170510120909121506060606101515090404090909142216191318101317051021
211509151506050912090910060509151008080805091518262527262924181827221218
180912120904121212040405060907090511061012190122445052494742302117191221
241215151212121212121206051579150915131420212546645660555048504731251718
150610151510080505091005011579120610070909123334652464441433941554631 1521
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210612121643514651190510101205151505121417263749543501102330476255221327
210409041949524954190509121001101512171513203552523205101024586553281721
240412151237514534170612150505081404171313123648483617212433466256281424
181212121215100115151012060508051306060505153441444016181821476546201424
240904071210050110151510050108131409121005153445503816080527565636171330
241210121501010107513161813080510101005051016335042341316092458512015153
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240409090505050925364751554120130105011018255053291608100509445937211830
180915150508052039546659656262928231009274556531917212116144357442111730
210906121008062750665642566848453629171328434938113132272509325752241330
210609120911234468695036506556535332131340504732194125334111416558211333
180912141835465969613825314245495838182458614922294322383611456451200527
211510133354615554583617213225406047213162501912343508544220617041120530
181210153958665445543518262825405241254063411015333015453330616732221530
211510154970563834342015293448575142173762440913272233533732676729212327
181210154468704732261730546672684730144151331523281634412040705932271830
211205103160716463482133617767542921284850391534181140461653736133231124
211510082034495359582434727136181320335048332434171749422461735929130127
180910051018334457664040707035050516405965403545132852323265705530080830
211215050510223354645244696128090914384965462738293259433461685319060630
241005010510121643584951595024171317294465614137292452524763603814081130
300101010101051526394043473120181521253550525040403443495658543409130824
270501010105090413242836292013150418182443464851414141464039392210131221
300501010105050526091722201513140912150817213341443126233139343010001051521
300801050508101413130812100605061005011318120917201212202315050101051524
270805100510010805080508081012121000171081014050813242413050105101005 0527
010101010101010101010101010101010101010101010101010101010101010101010101
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FIG. 3. Values for a brightness contrast function for the data in Fig. 1 using the function in Fig. 2.

which constitutes a blob for which all the corresponding points in the contrast function have magnitude less than or equal to K and having the property that any point not in the blob, but adjacent to a point of the blob, has a value for its contrast function greater than K. Intuitively speaking, a blob of heterogeneity K is simply a maximal size contiguous region in which the contrast function is less than or equal to K, bounded by points having contrast greater than K. It will be seen that for any value of K, a brightness function like Fig. 1 can be partitioned into (generally) disjoint subregions. Figure 9 shows the partitioning of Fig. 3 for $K = 40$.

The problem of doing a morphological analysis can be seen as one of choosing suitable values of K for the partitioning of a region like Fig. 1 into subregions. A general method for achieving such a partitioning may be based upon two observations. First, we note that whatever criterion is used for establishing the boundary of a region considered as figure should also be used for establishing the boundary of a region considered as ground. That is, the objects and their background should be treated uniformly by such a partitioning method. Secondly, we note that the boundaries of a region may be variously defined according to the degree of heterogeneity attributed to that region. But whereas two regions of heterogeneity K may be disjoint, they may nevertheless constitute parts of a single region of heterogeneity $K+1$. While there is no obvious criterion for generally requiring regions to have a given heterogeneity, it is possible to allow regions to be defined in such a way that they are defined by some maximum heterogeneity, such that any greater heterogeneity would lead to disjoint regions coalescing into single regions.

It can be seen that we have implicitly specified an algorithm for obtaining a set of different partitions of a brightness function like Fig. 1. The first partitioning is into all blobs of heterogeneity 0, the second partitioning is into all blobs of heterogeneity 1, etc. Furthermore, these partitionings induce a partial ordering represented by a tree structure on the original brightness function in a certain natural way. The nodes of such a tree are the blobs of heterogeneity K for each value of K ranging from 0 to some maximum, the maximum being the value of K for which the whole of Fig. 1 constitutes a single blob of heterogeneity K. Whenever two blobs of heterogeneity K coalesce into a single blob of heterogeneity $K+1$, we represent this by having the node for the single blob cover the node for the two constituent blobs in the tree. Figure 4 is an example of how such a tree structure might appear. Blobs A, B, C, D, E are blobs of heterogeneity 0; blobs G, L and J are blobs of heterogeneity 4, etc. Blobs H and I, which are disjoint at $K = 3$, coalesce into a single blob, L, at $K = 4$.

Such a complete morphological decomposition of an image is unnecessarily elaborate and may generally be reduced significantly if we notice (in the example) that the sequence of blobs from A through G in Fig. 4 provides very little additional information beyond that given by either A or G. We may thus consider reducing such a tree as the one in Fig. 4 to a considerably simpler one as shown in Fig. 5.

We should note that for real images the (homomorphic) tree reduction involved

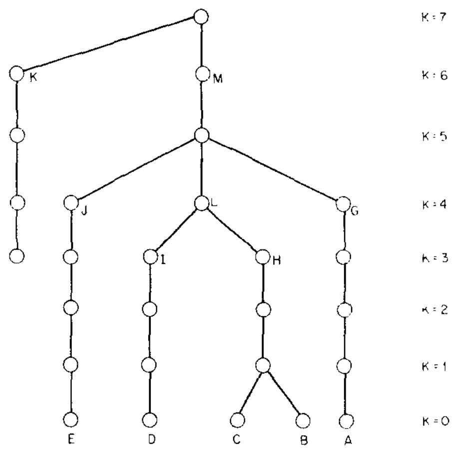

FIG. 4. The tree structure of a complete morphological decomposition of an image.

in going from Fig. 4 to Fig. 5 will nevertheless usually produce very large trees. Figure 6 is an example of such a reduced tree obtained from the image in Fig. 1 using the contrast function in Fig. 3. The tree is represented in Fig. 6 as a set of quintuples. For each quintuple the first pair of numbers represents the row and column coordinates of the upper left-hand corner of a blob, the number of points in which is given by the third number and whose degree of heterogeneity is given by

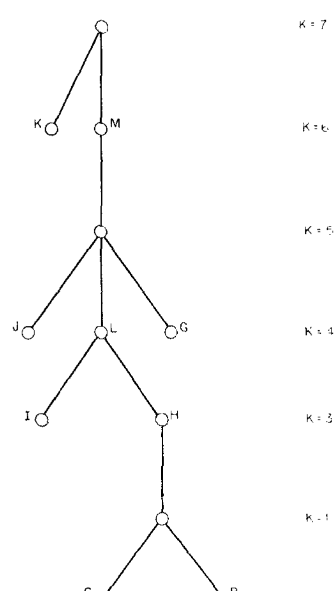

FIG. 5. A reduced tree structure for a morphological decomposition of an image.

the fourth number. Position in the tree is indicated by indentation in Fig. 6. Part of the structure given in Fig. 6 is shown in conventional tree form in Fig. 7, where

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(((1 1 1070 48 M)
   (19 12 41 48 M)
   ((1 1 857 40 M)
    (8 30 131 40 M)
    ((((((((1 1 665 20 M)
           (24 19 2 20 M)
           ((1 1 602 17 M)
            (6 33 1 17 M)
            (18 35 7 17 M)
            ((1 1 577 16 M)
             (8 35 3 16 M)
             (12 35 5 16 M)
             ((1 1 509 14 M)
              (27 17 7 14 M)
              (((1 1 465 12 M)
                (34 26 2 12 M)
                ((1 1 206 11 M)
                 (5 3 2 11 M)
                 (8 2 9 11 M)
                 (13 7 184 11 M)
                 ((1 1 145 9 M)
                  (12 18 17 9 M)
                  ((1 1 89 8 M)
                   (8 7 3 8 M)
                   (8 17 1 8 M)
                   (8 19 1 8 M)
                   (10 11 3 8 M)
                   (10 15 14 8 M)
                   (10 18 1 8 M)
                   ((1 1 62 7 M)
                    (4 12 16 7 M)
                    (6 17 1 7 M)
                    (6 19 1 7 M)
                    (((1 1 44 5 M)
                      (3 19 4 5 M)
                      (3 26 1 5 M)
                      (3 29 1 5 M)
                      (4 5 1 5 M)
                      (5 7 2 5 M) ((1 1 36 4 M) (3 34 1 4 M))))
                    (((5 14 1 5 M)
                      (6 10 7 5 M) ((6 10 2 4 M) (8 13 1 4 M)))))
                   (((((11 15 1 4 M) (14 14 1 4 M)))))))
                 (((8 2 5 8 M)
                   (11 4 1 8 M) (((8 2 1 6 M) (10 2 3 6 M)))))
                 ((13 8 31 9 M)
                  (14 2 3 9 M)
                  (26 3 106 9 M)
                  (27 35 8 9 M)
                  (28 2 1 9 M)
                  ((13 8 18 8 M) (17 2 1 8 M) (19 3 2 8 M))
                  ((26 3 101 8 M)
                   (32 17 1 8 M)
                   ((26 3 1 7 M)
                    (28 4 85 7 M)
                    (34 14 3 7 M)
                    ((((30 4 13 4 M)
                       (32 8 1 4 M)
                       (33 33 3 4 M) (34 19 41 4 M))))))
                  (((27 35 1 7 M) (29 34 2 7 M)))))))))))))))))
    ((9 26 110 34 M)
     (29 23 1 34 M)
     ((9 27 84 31 M)
      (25 29 3 31 M)
      ((9 27 74 28 M)
       (30 26 1 28 M)
       (((9 27 49 22 M)
         (21 27 10 22 M)
         ((9 27 40 21 M)
          (22 23 8 21 M)
          ((((9 27 6 16 M)
             (13 27 19 16 M)
             (19 25 1 16 M)
             (((14 28 10 13 M)
               (19 30 3 13 M)
               ((14 28 3 12 M)
                (16 27 5 12 M)
                (((17 27 1 9 M) (17 29 2 9 M))))))))))
         ((22 27 2 21 M)
          (25 26 6 21 M) ((((25 26 2 16 M) (28 25 1 16 M))))))))))
   ((((((21 12 11 27 M)
        (22 15 2 27 M) (((((22 12 1 17 M) (24 12 2 17 M)))))))))))
```

FIG. 6. Complete morphological decomposition of the image given by Fig. 1 with the contrast function in Fig. 3.

we see the whole original image is partitioned into two component blobs of heterogeneity 48, the first being located at coordinates 1, 1 and having an area of 1070, the second at coordinates 19, 12 with an area of 41. Each of these blobs, in turn, is

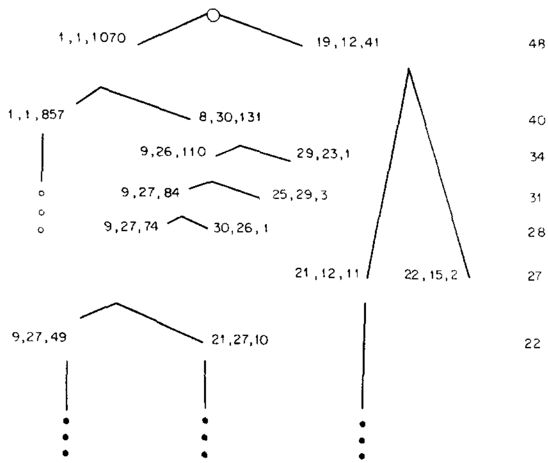

FIG. 7. Top of tree for reduced structure in Fig. 6.

partitioned into other subblobs where the heterogeneity of those blobs is shown in Fig. 7. Blob (8, 30, 131, 40) in Fig. 7 is the right-hand blob of Fig. 9 and blob (19, 12, 41, 48) in Fig. 7 is the left-hand blob in Fig. 10.

B. Output Form of Morphological Analysis

The type of structure presented in Figs. 6 and 7, which represent the decomposition of the original image in Fig. 1, actually consists of a whole class of different decompositions. To specify any particular decomposition, we must choose a set of nodes which constitutes a cover of the top node of the tree. Thus, in Fig. 5 such a cover would consist of the nodes *K* and *M* or *K* and *J*, *I*, *H*, *G*, or *K*, *J*, *L*, *G*, etc.

By selecting a lower set of nodes, as for example, *I* and *H* rather than *L* in Fig. 4, we thereby choose a more precise form of decomposition of the image structure than if we were to choose the higher node. The sets of lower nodes represent a close approximation to the notion of a refinement of a partitioning. Thus, the partition

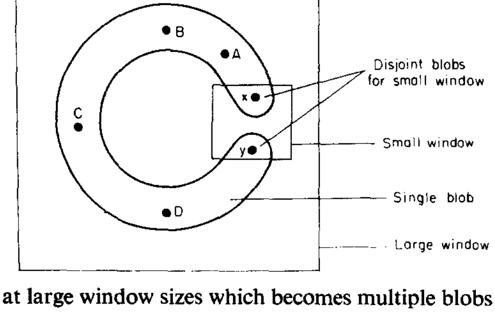

FIG. 8. A single blob at large window sizes which becomes multiple blobs at small window sizes.

into K, J, L, and G represents a refinement of the partitioning into K and M in Fig. 4. Actually this fails to be precisely the case of a refinement of a partitioning, insofar as the total area of blobs J, L and G is slightly less than that of blob M. The difference in the areas or number of points constituting these two sets of blobs is attributable to the points constituting the boundaries of the blobs. But, nevertheless, any such cover of the top node of the tree constitutes a proper morphologic

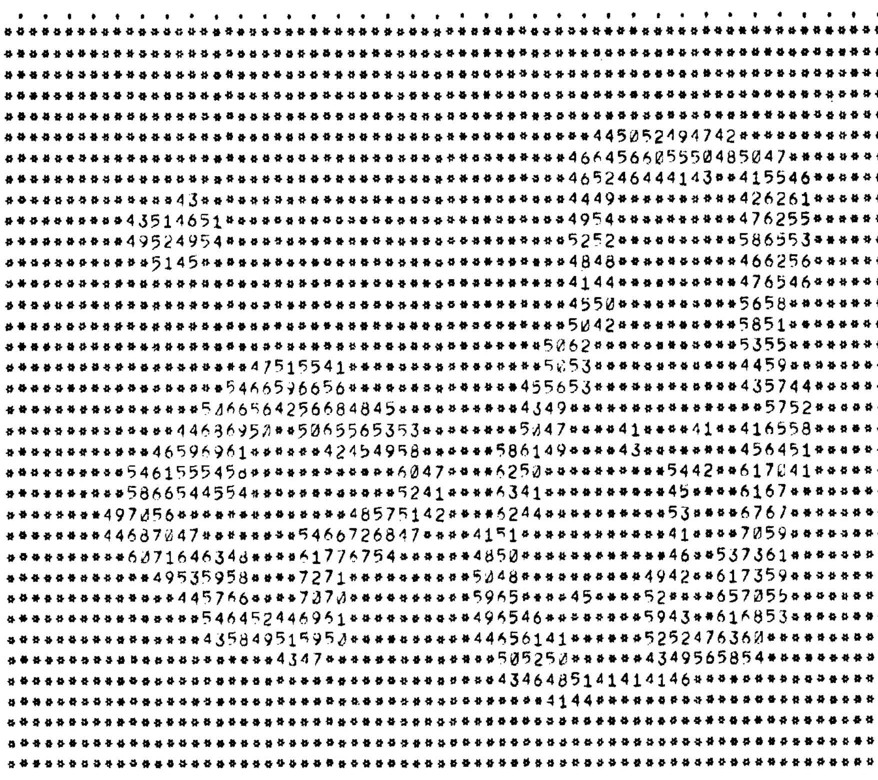

FIG. 9. Partitioning of the contrast function in Fig. 3 for $K = 40$. Note the presence of three blobs (one presumably being the background).

decomposition of the whole image. The question of the choice of such a particular cover must be deferred to higher stages in the structural analysis. These stages are the ones that make use of syntactic and even semantic information about the structure of the image.

There is an interesting interpretation that may be given to the process of choosing covers for a morphological decomposition tree. From the standpoint of morphological analysis, any cover is acceptable, but choosing a more refined cover, i.e., one using lower nodes in the tree, can be understood as the analog of the process of more precise measurement in more conventional types of measurement systems. Here we are dealing with images and their structure. The proper representation of an image's structure appears as a more complex object than does the representation of other types of scientific quantities which lend themselves more naturally to numerical measurement. In numerical measurement, when we proceed from a gross determination to a more refined determination of a quantity, we represent this

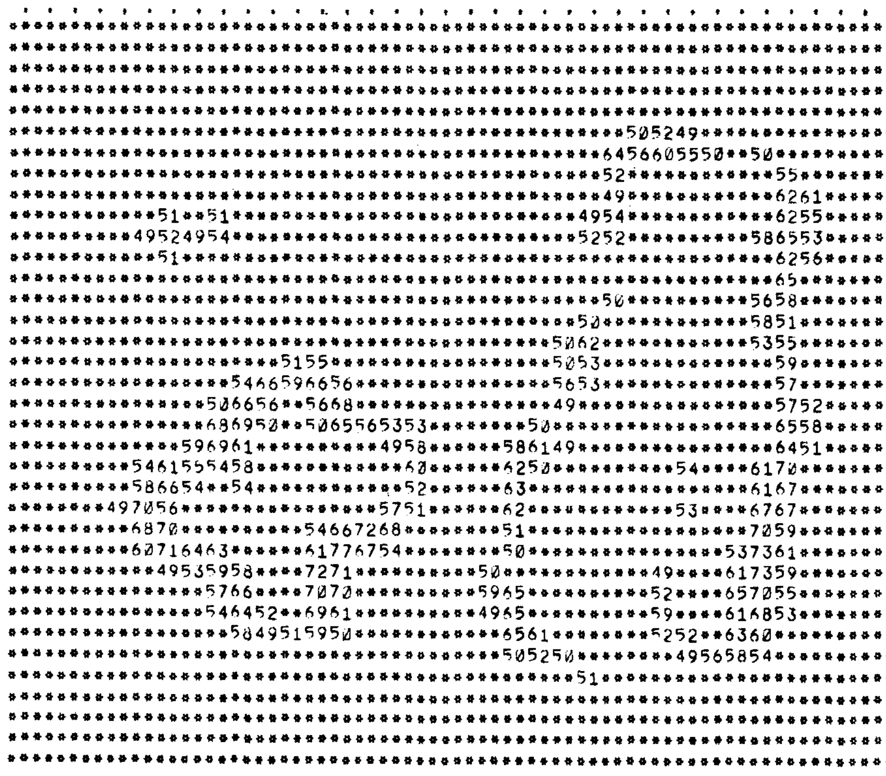

FIG. 10. Partitioning of the contrast function in Fig. 3 for $K = 48$. Note that the right hand of Fig. 9 has coalesced with the background by $K = 48$.

more refined determination by a number having more precision in its representation (more figures of significance). By analogy, when we describe the constituent structure of an image, we may elect to give more precise specification of this structure by choosing a more refined cover for the constituent structure tree.

It is useful to recognize that this process of increasing precision of representation of an image has many of the attributes of the more customary process of increasing the precision of specification of any other type of measurement. One would hope that as the science of mechanical determination of image structure progresses, some of the attributes of more conventional precise measurement will be seen to play an important role. What we are suggesting here is that a famous remark by Lord Kelvin (*5*) regarding numerical precision and its effects on scientific measurement be extended to include data structures more complex than the real number system for which mathematical operations are just as available, though less commonly understood, than they are for numerical measurements.

C. Generalizations of the Algorithm

The above discussion and the example presented are both intended to illustrate a particular instance of a class of decomposition algorithms. Several arbitrary choices have been made in the example above for purposes of clarity. We now wish to describe generalizations of this algorithm which yield slightly different decompositions of images. These generalizations have some consequences with respect to the complexity of processing on a computer, and may have consequences with respect to the usefulness of such an algorithm as a model for visual recognition and the psychophysics of visual perception.

Our first generalization is with respect to the source of numercial data to be processed. We suggested above that the numbers in the initial array represent some brightness function. The methods whereby such a brightness function may be assigned to a naturally occurring image are many and varied. A good survey of the psychophysical bases for assigning brightness functions as well as contrast measures, rather different from the one used above, can be found in the review by Brown and Mueller (*6*).

We have assumed above that the discrete numerical values associated with whatever brightness function has been chosen are assigned to elements in some discrete tessellation of the plane. For purposes of use in a decomposition algorithm, some such tessellation is required, but the implied rectangular one used above is not the only one which may conveniently be chosen. Others include a triangular tessellation in equilateral triangles which fill the plane, and another, which also leads to a simple scanner implementation, is a tessellation into regular hexagons. Of course, there is no necessary requirement that the tessellation be a regular one. A model in which the spacing of resolution elements varies as a function of position in the plane is also one that might be considered. Along with the nature of the tessellation chosen, there is also the matter of choosing a suitable topology, particularly that aspect concerned with the notion of adjacency. In the algorithm described above, we have implicitly assumed a connectivity in the plane such that an element is considered adjacent to eight neighboring elements. Other choices are possible; obviously the four neighbor choices and others which assume different degrees of connectivity for points located at different distances from a given element.

In the operation of the decomposition algorithm the notion of the degree of heterogeneity occurs, and we chose above a rather arbitrary resolution for this heterogeneity measure in terms of minimally resolvable units in the contrast function. These minimally resolvable units (more usually referred to as just noticeable differences) may be chosen in many different ways. Again the reader is referred to the psychophysics literature.

Another rather subtle generalization in our algorithm can be obtained if one looks more carefully at the two related notions of precision of representation and resolution of measurement. We have suggested above that the counterpart of ordinary numerical precision occurs in the choice of depth in a covering tree used to represent an image decomposition. The number of disjoint blobs that occur in such a cover is, in turn, a consequence of a choice that has been arbitrarily made in the above example. This choice involves the size of the "window" used for measuring disjointedness of two blobs. When two disjoint blobs come together, it is this coalescence that manifests itself in the appearance of those blobs as elements in the tree representation. Furthermore, if two regions in the original image are connected to each other by no shorter path than one which involves long distances in the original image, these two such regions will still be parts of the same blob. Consider a case such as that of Fig. 8. Here we see that points x and y are parts of the same blob not because there is any direct connection between x and y, but only because of a remote connection between points a, b, c, d, etc. One might imagine the use of a small viewing window, such that superimposing this window upon the original blob, in Fig. 8, will in all cases place points x and y in blobs disjoint within the small window. For blobs that are not convex, as in Fig. 8, a suitable choice of small window can cause a single blob to be treated as two or more separate blobs.

The window implicitly used in the above example, of course, is a window larger than the whole image. This corresponds to the use of a measuring tool whose resolution is very gross. Consequently, regions where certain kinds of fine structure "information" in the original image appear will not be resolved by so gross a measuring instrument. The use of a smaller window will enable the resolution of such information containing regions. The consequence of this observation is that the choice of a suitable size window for measuring connectedness of points or connectiveness of regions in an original image determines the resolution with which the image can be decomposed into its constituent parts. Then the depth in the tree, with which the image is partitioned, represents precision of the image's description. One would expect that good measurement practice dictates the use of high precision in representation only in the cases where high resolution is available in the measuring instrument. Thus, with small windows we choose covers for the representing tree which are far down in the tree. For large windows we choose higher covers nearer the top node of the tree.

All of the above described modifications and generalizations of the algorithm we have discussed lead to slightly different kinds of decompositions. The notion that remains basic, however, is that of taking an image and decomposing it in a variety of different ways into constituent parts, based on information which is intrinsic to the image. The tests for validity of such a decomposition are twofold:

first, whether such a morphological decomposition can profitably be used at a higher syntactic level where knowledge of the structure of the information source is imposed from outside, and second, whether certain practical applications can be achieved using decomposition to detect subobjects in an image, and to be able to resynthesize the image from such a decomposition.

ACKNOWLEDGMENTS

The algorithm described here was written by the author in the LISP language on the ANFSQ-32 computer, where the experimental results were obtained. Some versions of the algorithm were so complex that no practical results would have been obtained were it not for the great speed of operation and elegance of basic assembly language code written by Mrs. Ida Rhodes. The numerical data for the examples were converted by Miss Anne Holston on the PDP-10 computer from data presented in Mendelsohn's paper. A subsequent version of this algorithm was built by Richard Feldmann on the PDP-10 computer. It should be no surprise to his colleagues that it runs twenty times faster than the author's version. Quite independently of the above effort, a very similar algorithm has been created by L. Krakauer at M.I.T. which will be reported in his dissertation.

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